Math 304 Midterm 2 Sample

Name: ____________________

This exam has 9 questions, for a total of 100 points.

Please answer each question in the space provided. You need to write **full solutions**. Answers without justification will not be graded. Cross out anything the grader should ignore and circle or box the final answer.

Question	Points	Score
1	12	
2	5	
3	10	
4	10	
5	15	
6	12	
7	10	
8	16	
9	10	
Total:	100	

Question 1. (12 pts)

Determine whether each of the following statements is true or false. You do NOT need to explain.

(a) If A is an $m \times n$ matrix, then A and A^T have the same rank.

Solution: True

(b) Given two matrices A and B, if B is row equivalent to A, then B and A have the same row space.

Solution: True.

(c) Given two vector spaces, suppose $L : V \to W$ is a linear transformation. If S is a subspace of V, then $L(S)$ is a subspace of W.

Solution: True.

(d) For a homogeneous system of rank r and with n unknowns, the dimension of the solution space is $n - r$.

Solution: True.

Question 2. (5 pts)

Find the angle between $v = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $w = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ in $\mathbb{R}^3$.

Solution: We denote the angle between v and w by θ.

$$\cos\theta = \frac{< v, w >}{\|v\|\|w\|} = \frac{1}{\sqrt{2}}$$

So

$$\theta = \arccos(\frac{1}{\sqrt{2}})$$

(In fact, in this case, we know $\theta = \pi/4$).

Question 3. (10 pts)

Let V be the subspace of $\mathbb{R}^3$ spanned by $v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Describe $V^\perp$ by finding a basis of $V^\perp$.

Solution: $V^\perp$ consists of vectors (a, b, c) which satisfy the following condition.

$$(1, 1, 1) \cdot (a, b, c) = 0$$

That is,

$$a + b + c = 0$$

So we have

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = s \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

It follows that

$$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

form a basis of $V^\perp$.

Question 4. (10 pts)

Given two lines

$$L_1 : x = t + 1,\ y = 3t + 1,\ z = 2t - 1,$$

and

$$L_2 : x = 2t - 2,\ y = 2t + 3,\ z = t + 1,$$

suppose a plane H is parallel to both L_1 and L_2. Moreover, H passes through the point $(0, 1, 0)$. Find the equation of H.

Solution: A normal vector of H is orthogonal to the direction vector of L_1: $u = (1, 3, 2)$ and the direction vector of L_2: $v = (2, 2, 1)$.

Calculate the cross product of u and v:

$$u \times v = (-1, 3, -4)$$

So the equation of H is

$$-x + 3y - 4z = 3$$

Question 5. (15 pts)
Given

$$A = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 1 \end{bmatrix}$$

(a) Find a basis of Ker(A).

Solution: First, use elementary row operations to get a row echelon form of A.

$$\begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So all elements in Ker$A$ are of the form

$$t \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -4 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

So

$$v_1 = \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

form a basis of the kernel.

(b) Find a basis of the row space of A.

Solution: The two nonzero rows in the row echelon form of A form a basis of the row space of A. That is

$$u_1 = (1, 0, 2, 4)$$

$$u_2 = (0, 1, -3, -1)$$

form a basis of the row space of A.

(c) Find a basis of the range of A.

Solution: Note that the range of A is the same as the column space of A. Use the row echelon from the part (a), we see that the 1st and 2nd of A form a basis of the range of A. That is,

$$w_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}, w_2 = \begin{bmatrix} 0 \\ 1 \\ 4 \\ -1 \end{bmatrix}$$

form a basis of the range of A.

(d) Determine the rank of A.

Solution: The rank of A is the dimension of the row space. So the rank of A is 2. (In fact, we know that the rank of A is also the same as the dimension of the column space of A.)

Question 6. (12 pts)

Determine whether the following mappings are linear transformations.

(a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 1 \\ x_1 + x_2 \end{pmatrix}$$

Solution: T is not a linear transformation. For example,

$$T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

on the other hand,

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} + T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

So

$$T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \neq T\begin{pmatrix} 1 \\ 0 \end{pmatrix} + T\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(b) $L : \mathbb{P}_2 \to \mathbb{P}_2$ by

$$L(p(x)) = p'(x) + p(x)$$

Solution: Suppose $p_1(x), p_2(x) \in \mathbb{P}_2$ and $\alpha, \beta \in \mathbb{R}$.

$$\begin{aligned} L(\alpha p_1(x) + \beta p_2(x)) &= \alpha p_1'(x) + \beta p_2'(x) + \alpha p_1(x) + \beta p_2(x) \\ &= \alpha L(p_1(x)) + \beta L(p_2(x)) \end{aligned}$$

So L is a linear transformation.

Question 7. (10 pts)

Let $M_2(\mathbb{R})$ be the space of all (2×2) matrices with real coefficients. The set

$$S = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

is a basis of $M_2(\mathbb{R})$. Find the coordinates of $A = \begin{pmatrix} 5 & 3 \\ 3 & 1 \end{pmatrix}$ with respect to the basis S.

Solution: We need to write

$$\begin{pmatrix} 5 & 3 \\ 3 & 1 \end{pmatrix} = a_1 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + a_2 \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} + a_3 \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} + a_4 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

That is, we need to solve the linear system

$$\begin{cases} a_1 + a_2 + a_3 + a_4 = 5 \\ a_1 - a_2 - a_3 = 3 \\ a_1 + a_2 = 3 \\ a_1 = 1 \end{cases}$$

Simply use back substitution. We have

$$a_1 = 1, a_2 = 2, a_3 = -4, a_4 = 6$$

So

$$[A]_S = \begin{bmatrix} 1 \\ 2 \\ -4 \\ 6 \end{bmatrix}$$

Question 8. (16 pts)

Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation given by

$$L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2 \\ x_1 - x_2 \end{pmatrix}$$

(a) Find the matrix representation of L with respect to the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$.

Solution:

$$L(e_1) = L\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$L(e_2) = L\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

So the matrix representation of L with respect to the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$ is

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & -1 \end{pmatrix}$$

(b) Let $u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $u_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Moreover, let $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Find the matrix representation of L with respect to the basis $\{u_1, u_2\}$ of $\mathbb{R}^2$ and the basis $\{v_1, v_2, v_3\}$ of $\mathbb{R}^3$.

Solution:

$$L(u_1) = L\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$L(u_2) = L\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

Now we need to find the coordinate vectors of $L(u_1)$ nad $L(u_2)$ with respect to the basis $B = \{v_1, v_2, v_3\}$ of $\mathbb{R}^3$.

$$L(u_1) = av_1 + bv_2 + cv_3$$

Solve for a, b, c, and we get

$$[L(u_1)]_B = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Similarly, we obtain

$$[L(u_2)]_B = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

So the matrix representation of L in this case is

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 0 \end{pmatrix}$$

Question 9. (10 pts)

Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Its matrix representation with respect to the standard basis of $\mathbb{R}^2$ is

$$\begin{pmatrix} -2 & 2 \\ -6 & 5 \end{pmatrix}.$$

(a) Find the transition matrix from the basis $\{u_1, u_2\}$ to the standard basis $\{e_1, e_2\}$, where

$$u_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, u_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Solution: The transition matrix is

$$U = \begin{pmatrix} | & | \\ u_1 & u_2 \\ | & | \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$$

(b) Find the matrix representation of L with respect to $\{u_1, u_2\}$.

Solution: The matrix representation of L with respect to $\{u_1, u_2\}$ is

$$U^{-1} \begin{pmatrix} -2 & 2 \\ -6 & 5 \end{pmatrix} U,$$

where U is the matrix from part (a). In this case, an easy way to find U^{-1} is

$$U^{-1} = \frac{1}{\det U} \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

So

$$U^{-1} \begin{pmatrix} -2 & 2 \\ -6 & 5 \end{pmatrix} U = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} -2 & 2 \\ -6 & 5 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$